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1994 J. Phys. A: Math. Gen. 27 4301

(http://iopscience.iop.org/0305-4470/27/12/033)

J. Phys. A: Math. Gen. 27 (1994) 4301–4315. Printed in the UK

The Duffin–Kemmer–Petiau oscillator

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Received 17 January 1994

Abstract. In view of the current interest in relativistic spin-1 systems and the recent work on the Dirac oscillator, we introduce the Duffin–Kemmer–Petiau (DKP) equation obtained by using an external potential linear in r. Since the spin-0 representation leads to a harmonic oscillator in the non-relativistic limit and becomes an harmonic oscillator with a spin-orbit coupling of the Thomas form for vector bosons, we call the equation the DKP oscillator. This oscillator is a relativistic generalization of the quantum harmonic oscillator for scalar and vector bosons. We show that it conserves total angular momentum and that it is *exactly* solvable for both scalar and vector DKP bosons. We calculate and discuss the eigenvalues and eigenstates of the DKP oscillator in the spin-0 and spin-1 representations.

1. Introduction

The theory of the harmonic oscillator is important in physics since it enters all problems involving quantized oscillations. It has wide applications for systems with linear and quasilinear equations of motion.

There has been a great deal of interest recently in the Dirac oscillator [1–6]. It was shown that the Dirac oscillator, whose Hamiltonian is linear in r, conserves angular momentum, is exactly solvable and its eigenspectrum is highly degenerate. The interest in the Dirac oscillator is mainly motivated by using it as a quark-confining potential in QCD and also as a suitable analytic basis to deal with more realistic interactions.

In view of the current interest in relativistic spin-1 systems [7–10], it is the aim of this paper to analyse the Duffin–Kemmer–Petiau (DKP) oscillator in order to provide an analytic and mathematically simple basis with which more complex interactions can be studied.

Unlike other relativistic wave equations for bosons, the DKP one is a first-order equation [11]. We therefore introduce a system obtained from the free DKP equation by a non-minimal substitution linear in r. Since its spin-0 representation leads to a harmonic oscillator in the non-relativistic limit and it becomes a harmonic oscillator with a spin-orbit coupling of the Thomas form for vector bosons, we call the system a DKP oscillator. This system is a relativistic generalization of the quantum harmonic oscillator for scalar and vector particles. We demonstrate that it conserves total angular momentum and that it is exactly solvable for both scalar and vector DKP bosons. We also compute and discuss the eigenvalues and eigenstates of the DKP oscillator in both cases.

2. The DKP oscillator

For a free scalar or vector boson of mass m, the relativistic DKP equation is

$$(c\beta \cdot p + mc^2)\psi = i\hbar\beta^0 \frac{\partial \psi}{\partial t} \tag{1}$$

0305-4470/94/124301+15$19.50

where the internal variables β^μ ($\mu = 0, 1, 2, 3$) satisfy the commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu. \tag{2}$$

In the spin-0 representation, β^μ are 5×5 matrices defined as

$$\beta^0 = \begin{pmatrix} \vartheta & \tilde{0} \\ \tilde{0}_T & \mathbf{0} \end{pmatrix} \qquad \beta^i = \begin{pmatrix} \hat{0} & \rho^i \\ -\rho^i_T & \mathbf{0} \end{pmatrix} \qquad i = 1, 2, 3 \tag{3a}$$

with $\hat{0}$, $\tilde{0}$, $\mathbf{0}$ as 2×2, 2×3, 3×3 zero matrices, respectively, and

$$\vartheta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3b}$$

while the dynamical state ψ is a five-component spinor. For vector bosons, ψ is a 10-component spinor and β^μ are 10×10 matrices given by

$$\beta^0 = \begin{pmatrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \overline{0}^T & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \beta^i = \begin{pmatrix} 0 & \overline{0} & e_i & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & -\mathrm{i}s_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & -\mathrm{i}s_i & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4a}$$

where s_i are the usual 3×3 spin-1 matrices

$$\overline{0} = (0 \quad 0 \quad 0) \qquad e_1 = (1 \quad 0 \quad 0) \qquad e_2 = (0 \quad 1 \quad 0) \qquad e_3 = (0 \quad 0 \quad 1) \tag{4b}$$

while $\mathbf{I}$ and $\mathbf{0}$ designate the 3×3 identity and zero matrices, respectively.

For the external potential which we introduce with the non-minimal substitution

$$\boldsymbol{p} \to \boldsymbol{p} - \mathrm{i}m\omega\eta^0\boldsymbol{r} \tag{5}$$

where ω is the oscillator frequency and $\eta^0 = 2\beta^{0^2} - 1$, the DKP equation for the system is

$$\left[c\boldsymbol{\beta} \cdot (\boldsymbol{p} - \mathrm{i}m\omega\eta^0\boldsymbol{r}) + mc^2\right]\psi = \mathrm{i}\hbar\beta^0\frac{\partial\psi}{\partial t}. \tag{6}$$

This external potential, which is of Lorentz-tensor type, does not conserve the orbital and spin angular momenta, since

$$[\boldsymbol{\beta}\eta^0 \cdot \boldsymbol{r}, \boldsymbol{L}] = -\mathrm{i}(\boldsymbol{\beta}\eta^0 \wedge \boldsymbol{r}) \qquad \text{and} \qquad [\boldsymbol{\beta}\eta^0 \cdot \boldsymbol{r}, \boldsymbol{S}] = \mathrm{i}(\boldsymbol{\beta}\eta^0 \wedge \boldsymbol{r}) \tag{7}$$

but it does conserve the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

In the spin-0 representation, we set [12]

$$\psi(\boldsymbol{r}) = \begin{pmatrix} \psi_{\text{upper}} \\ \mathrm{i}\psi_{\text{lower}} \end{pmatrix} \qquad \text{with} \quad \psi_{\text{upper}} \equiv \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \qquad \text{and} \quad \psi_{\text{lower}} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{8}$$

so that for stationary states the DKP equation can be rewritten as

$$\begin{cases} mc^2\phi = E\varphi + \mathrm{i}c(\boldsymbol{p} + \mathrm{i}m\omega\boldsymbol{r}) \cdot \boldsymbol{A} \\ mc^2\varphi = E\phi \\ mc^2\boldsymbol{A} = \mathrm{i}c(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r})\phi. \end{cases} \tag{9}$$

Eliminating φ and $\boldsymbol{A}$ in favour of ϕ yields the Klein–Gordon equivalent equation

$$(E^2 - m^2c^4)\phi = \left[c^2(p^2 + m^2\omega^2r^2) - 3\hbar\omega mc^2\right]\phi. \tag{10}$$

Using the relation $E = \varepsilon + mc^2$ and the non-relativistic limit $\varepsilon \ll mc^2$ transforms equation (10) into

$$\varepsilon\phi = \left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2r^2 - \frac{3}{2}\hbar\omega\right]\phi \tag{11}$$

which describes the traditional isotropic harmonic oscillator.

In the spin-1 representation of equation (6), the dynamical state ψ is the 10-component spinor [13]

$$\psi(\boldsymbol{r}) = \begin{pmatrix} \mathrm{i}\varphi \\ \boldsymbol{A}(\boldsymbol{r}) \\ \boldsymbol{B}(\boldsymbol{r}) \\ \boldsymbol{C}(\boldsymbol{r}) \end{pmatrix} \quad \text{with} \quad \boldsymbol{A} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \boldsymbol{B} \equiv \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} \equiv \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \tag{12}$$

so that, for stationary states, the equation of motion equation (6) decomposes into

$$\begin{cases} mc^2\varphi = \mathrm{i}c\boldsymbol{p}^- \cdot \boldsymbol{B} & (13a) \\ mc^2\boldsymbol{A} = E\boldsymbol{B} - c\boldsymbol{p}^+ \wedge \boldsymbol{C} & (13b) \\ mc^2\boldsymbol{B} = E\boldsymbol{A} + \mathrm{i}c\boldsymbol{p}^+\varphi & (13c) \\ mc^2\boldsymbol{C} = -c\boldsymbol{p}^- \wedge \boldsymbol{A} & (13d) \end{cases}$$

where $\boldsymbol{p}^{\pm} = \boldsymbol{p} \pm \mathrm{i}m\omega\boldsymbol{r}$. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are the 3-component spinors analogous to the Dirac upper and lower components, respectively, we seek the wave equation for $\boldsymbol{A}$. Using equations (13*a*–*d*), it is straightforward to eliminate φ, $\boldsymbol{B}$ and $\boldsymbol{C}$ in favour of $\boldsymbol{A}$, so that one gets

$$\begin{aligned}(E^2 - m^2c^4)\boldsymbol{A} = {}& -c^2\boldsymbol{p}^+ \wedge (\boldsymbol{p}^- \wedge \boldsymbol{A}) + c^2\boldsymbol{p}^+(\boldsymbol{p}^- \cdot \boldsymbol{A}) \\ & - \frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^- \cdot [\boldsymbol{p}^+ \wedge (\boldsymbol{p}^- \wedge \boldsymbol{A})]\}. \end{aligned} \tag{14}$$

Evaluating the first two terms on the right-hand side of equation (14) (see appendix) yields

$$\begin{aligned}(E^2 - m^2c^4)\boldsymbol{A} = {}& [c^2(p^2 + m^2\omega^2r^2) - 3\hbar\omega mc^2 - 2\hbar\omega mc^2\boldsymbol{L} \cdot \boldsymbol{s}]\boldsymbol{A} \\ & - \frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^- \cdot [\boldsymbol{p}^+ \wedge (\boldsymbol{p}^- \wedge \boldsymbol{A})]\} \end{aligned} \tag{15}$$

where $\boldsymbol{L}$ is the orbital angular momentum and $\boldsymbol{s}$ the 3×3 spin-1 operator. In the non-relativistic limit $\varepsilon \ll mc^2$, the third term in equation (14) becomes negligible, since it is of order $1/m^3$, so that the wave equation for $\boldsymbol{A}$ can be written

$$\varepsilon\boldsymbol{A} \simeq \left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2r^2 - \frac{3}{2}\hbar\omega - \hbar\omega\boldsymbol{L} \cdot \boldsymbol{s}\right]\boldsymbol{A} \tag{16}$$

which characterises the usual harmonic oscillator in addition to a spin-orbit coupling, absent for scalar DKP bosons, of strength $-\hbar\omega$. Note that the strength of this coupling is one half of that obtained from the Dirac oscillator[1].

Since both the spin-0 and spin-1 representations of equation (6) lead to the usual three-dimensional (3D) oscillator, in the non-relativistic limit we refer to the system it describes as the Duffin–Kemmer–Petiau oscillator.

3. Solution of the scalar DKP oscillator problem

We now seek an exact solution to the $S = 0$ DKP oscillator eigenstate problem. In [12], we showed that the most general eigensolution for a central-field problem is

$$\psi_{JM}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} F_{nJ}(r)Y_{JM}(\Omega) \\ G_{nJ}(r)Y_{JM}(\Omega) \\ \mathrm{i}\sum_L H_{nJL}(r)\boldsymbol{Y}^M_{JL1}(\Omega) \end{pmatrix} \tag{17}$$

where $\boldsymbol{Y}^M_{JL1}(\Omega)$ are the normalized vector spherical harmonics while $F_{nJ}(r)$, $G_{nJ}(r)$ and $H_{nJ}(r)$ are radial wavefunctions. Inserting $\psi_{JM}(\boldsymbol{r})$, which is of parity $(-1)^J$, into equation (6) while setting $\alpha_J = \sqrt{(J+1)/(2J+1)}$, $\zeta_J = \sqrt{J/(2J+1)}$ and

$$F_{nJ}(r) = F(r) \qquad G_{nJ}(r) = G(r) \qquad H_{nJJ\pm1}(r) = H_{\pm1}(r) \tag{18}$$

yields the following set of first-order coupled radial differential equations

$$EF = mc^2 G \tag{19a}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} + \frac{m\omega r}{\hbar}\right)F = -\frac{1}{\alpha_J}mc^2 H_1 \tag{19b}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} + \frac{m\omega r}{\hbar}\right)F = \frac{1}{\zeta_J}mc^2 H_{-1} \tag{19c}$$

$$-\alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} - \frac{m\omega r}{\hbar}\right)H_1 + \zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} - \frac{m\omega r}{\hbar}\right)H_{-1} = \frac{1}{\hbar c}\left(mc^2F - EG\right). \tag{19d}$$

For the harmonic oscillator case, the coupling between the radial equations is simple so that inserting equation (19*a*–*c*) into (19*d*) leads to the homogeneous second-order differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(E^2 - m^2c^4)}{(\hbar c)^2} + \frac{3m\omega}{\hbar} - \frac{m^2\omega^2r^2}{\hbar^2} - \frac{J(J+1)}{r^2}\right)F(r) = 0. \tag{20}$$

This equation has the same form as the radial equation of the 3D harmonic oscillator and it is straightforward to show that its eigenvalues $E_{N,J}$ are

$$\frac{1}{2mc^2}(E^2_{N,J} - m^2c^4) = N\hbar\omega \tag{21}$$

where N is the principal quantum number defined as $N = 2n + J$, n being a non-negative integer representing the radial quantum number. The energy levels are equidistant and degenerate. For identification purposes, in what follows, it is $(E^2_{N,J} - m^2c^4)/2mc^2$ rather than $E_{N,J}$ to which we refer as energy levels, since the first form reduces to the usual Schrödinger eigenvalue in the non-relativistic limit. The corresponding eigenfunctions are

$$F_{N,J} = \lambda_{\mathrm{norm}}\exp\left(-\frac{m\omega}{2\hbar}r^2\right)\left(\frac{m\omega}{\hbar}r\right)^{J+1} L_n^{(J+\frac{1}{2})}\left(\frac{m\omega}{\hbar}r^2\right) \tag{22}$$

where $L_n^{(J+\frac{1}{2})}$ is the associated Laguerre polynomial of order n and λ_{norm} is a normalization constant. The remaining radial wavefunctions are

$$G = \lambda_{\text{norm}} \frac{E}{mc^2} e^{-(m\omega/2\hbar)r^2} \left(\frac{m\omega}{\hbar} r\right)^{J+1} L_n^{(J+\frac{1}{2})} \tag{23}$$

$$H_1 = -2\lambda_{\text{norm}} \alpha_J \frac{\hbar\omega}{mc^2} e^{-(m\omega/2\hbar)r^2} \left(\frac{m\omega}{\hbar} r\right)^{J} \left(nL_n^{(J+\frac{1}{2})} - (n + J + \tfrac{1}{2}) L_{n-1}^{(J+\frac{1}{2})}\right) \tag{24}$$

$$H_{-1} = 2\lambda_{\text{norm}} \zeta_J \frac{\hbar\omega}{mc^2} e^{-(m\omega/2\hbar)r^2} \left(\frac{m\omega}{\hbar} r\right)^{J} \left((2n + 2J + 1) L_n^{(J+\frac{1}{2})} - (n + J + \tfrac{1}{2}) L_{n-1}^{(J+\frac{1}{2})}\right). \tag{25}$$

The normalization constant λ_{norm} can be computed using the orthonormalization condition

$$\int_0^\infty \text{Re}\left[F^*(r) G(r)\right] \text{d}r = \tfrac{1}{2}.$$

4. Solution to the vector DKP oscillator problem

For the $S = 1$ central-field problem, the general eigenfunction we use takes the form [13]

$$\psi_{JM}(\boldsymbol{r}) = \frac{1}{r} \begin{pmatrix} \text{i}\phi_{nJ}(r) Y_{JM}(\Omega) \\ \sum_L F_{nJL}(r) \boldsymbol{Y}_{JL1}^M(\Omega) \\ \sum_L G_{nJL}(r) \boldsymbol{Y}_{JL1}^M(\Omega) \\ \sum_L H_{nJL}(r) \boldsymbol{Y}_{JL1}^M(\Omega) \end{pmatrix}. \tag{26}$$

Putting ψ_{JM} into equation (6) results in ten coupled radial differential equations which can be decoupled into two sets associated with the $(-1)^J$ and $(-1)^{J+1}$ parities. We call the $(-1)^J$ solutions natural-parity (or magnetic-like) states while we refer to the $(-1)^{J+1}$ solutions as unnatural-parity (or electric-like) states [13]. With the notation

$$R_{nJJ}(r) = R_0 \qquad R_{nJJ\pm1}(r) = R_{\pm1} \qquad R \equiv F, G, H \tag{27}$$

the set associated with the $(-1)^J$ parity is

$$EF_0 = mc^2 G_0 \tag{28a}$$

$$\hbar c \left(\frac{\text{d}}{\text{d}r} - \frac{J+1}{r} + \frac{m\omega r}{\hbar}\right) F_0 = -\frac{1}{\zeta_J} mc^2 H_1 \tag{28b}$$

$$\hbar c \left(\frac{\text{d}}{\text{d}r} + \frac{J}{r} + \frac{m\omega r}{\hbar}\right) F_0 = -\frac{1}{\alpha_J} mc^2 H_{-1} \tag{28c}$$

$$-\zeta_J \left(\frac{\text{d}}{\text{d}r} + \frac{J+1}{r} - \frac{m\omega r}{\hbar}\right) H_1 - \alpha_J \left(\frac{\text{d}}{\text{d}r} - \frac{J}{r} - \frac{m\omega r}{\hbar}\right) H_{-1} = \frac{1}{\hbar c} \left(mc^2 F_0 - EG_0\right). \tag{28d}$$

For unnatural-parity states, the radial differential equations are coupled in the following way :

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{J+1}{r}-\frac{m\omega r}{\hbar}\right)H_0=-\frac{1}{\zeta_J}\left(mc^2F_1-EG_1\right) \tag{29a}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{J}{r}-\frac{m\omega r}{\hbar}\right)H_0=-\frac{1}{\alpha_J}\left(mc^2F_{-1}-EG_{-1}\right) \tag{29b}$$

$$-\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{J+1}{r}+\frac{m\omega r}{\hbar}\right)F_1-\alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{J}{r}+\frac{m\omega r}{\hbar}\right)F_{-1}=\frac{1}{\hbar c}mc^2H_0 \tag{29c}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{J+1}{r}-\frac{m\omega r}{\hbar}\right)\phi=-\frac{1}{\alpha_J}\left(mc^2G_1-EF_1\right) \tag{29d}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{J}{r}-\frac{m\omega r}{\hbar}\right)\phi=\frac{1}{\zeta_J}\left(mc^2G_{-1}-EF_{-1}\right) \tag{29e}$$

$$-\alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{J+1}{r}+\frac{m\omega r}{\hbar}\right)G_1+\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{J}{r}+\frac{m\omega r}{\hbar}\right)G_{-1}=\frac{1}{\hbar c}mc^2\phi. \tag{29f}$$

To obtain the exact solution for the magnetic-like states, we eliminate G_0, $H_\pm$ in favour of F_0 in equation (28*d*). This yields the homogeneous second-order differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{(E^2-m^2c^4)}{(\hbar c)^2}+\frac{m\omega}{\hbar}-\frac{m^2\omega^2r^2}{\hbar^2}-\frac{J(J+1)}{r^2}\right)F_0(r)=0 \tag{30}$$

which is similar to the three-dimensional non-relativistic oscillator Schrödinger equation. The eigenvalues are

$$\frac{1}{2mc^2}(E_{N,J}^2-m^2c^4)=(N+1)\hbar\omega \tag{31}$$

with the principal quantum number $N=2n+J$ (n is the radial quantum number). Note that the oscillator levels are equidistant and degenerate; the zero-point energy differs here from that found for the scalar DKP bosons. The associated eigenfunctions are

$$F_0=\lambda_{\mathrm{norm}}\exp\left(-\frac{m\omega}{2\hbar}r^2\right)\left(\frac{m\omega}{\hbar}r\right)^{J+1}L_n^{(J+\frac{1}{2})}\frac{m\omega}{\hbar}r^2. \tag{32}$$

Using F_0, the remaining radial components of the DKP spinor can be trivially deduced from equations (28*a*–*c*).

To proceed with the exact solutions of the radial equations associated with unnatural-parity states, equations (29*a*–*f*) are transformed into

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{(E^2-m^2c^4)}{(\hbar c)^2}-\frac{3m\omega}{\hbar}-\frac{m^2\omega^2r^2}{\hbar^2}-\frac{J(J+1)}{r^2}\right)\phi=2\sqrt{J(J+1)}E\omega H_0 \tag{33a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{(E^2-m^2c^4)}{(\hbar c)^2}-\frac{m\omega}{\hbar}-\frac{m^2\omega^2r^2}{\hbar^2}-\frac{J(J+1)}{r^2}\right)H_0=2\sqrt{J(J+1)}E\omega\phi \tag{33b}$$

$$\begin{pmatrix}F_1\\G_1\end{pmatrix}=\frac{1}{E^2-m^2c^4}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{J+1}{r}-\frac{m\omega r}{\hbar}\right)\begin{pmatrix}\alpha_JE & \zeta_Jmc^2\\ \alpha_Jmc^2 & \zeta_JE\end{pmatrix}\begin{pmatrix}\phi\\H_0\end{pmatrix} \tag{33c}$$

$$\begin{pmatrix}F_{-1}\\G_{-1}\end{pmatrix}=\frac{1}{E^2-m^2c^4}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{J}{r}-\frac{m\omega r}{\hbar}\right)\begin{pmatrix}-\zeta_JE & \alpha_Jmc^2\\ -\zeta_Jmc^2 & \alpha_JE\end{pmatrix}\begin{pmatrix}\phi\\H_0\end{pmatrix}. \tag{33d}$$

The problem looks much simpler now since equations (33*a*–*b*) describe two coupled three-dimensional oscillator equations; once ϕ and H_0 are determined, the remaining radial wavefunctions can be deduced in a straightforward way using equations (33*c*–*d*).

It is worth noting that the two coupled second-order differential equations manifestly decouple for $J = 0$. In this case, one has to solve the trivial problem of two decoupled one-dimensional harmonic oscillators.

For the unnatural parity states with $J > 0$, these two equations can also be solved exactly. This can be done by the diagonalization procedure

$$\begin{pmatrix} \phi \\ H_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\gamma & \kappa \\ \kappa & -1-\gamma \end{pmatrix}\begin{pmatrix} R_+ \\ R_- \end{pmatrix} \qquad \text{with} \quad \gamma = \sqrt{1+\kappa^2} \quad \text{and} \quad \kappa = 2\sqrt{J(J+1)}\frac{E}{mc^2} \tag{34}$$

which decouples equations (33*a*–*b*) into the following radial equations:

$$\frac{\mathrm{d}^2 R_+}{\mathrm{d}r^2} + \left(\frac{E^2 - m^2c^4}{(\hbar c)^2} - 2\frac{m\omega}{\hbar} + \frac{\omega}{\hbar c^2}\sqrt{m^2c^4 + 4J(J+1)E^2} - \frac{m^2\omega^2r^2}{\hbar^2} - \frac{J(J+1)}{r^2}\right)R_+ = 0 \tag{35a}$$

$$\frac{\mathrm{d}^2 R_-}{\mathrm{d}r^2} + \left(\frac{E^2 - m^2c^4}{(\hbar c)^2} - 2\frac{m\omega}{\hbar} - \frac{\omega}{\hbar c^2}\sqrt{m^2c^4 + 4J(J+1)E^2} - \frac{m^2\omega^2r^2}{\hbar^2} - \frac{J(J+1)}{r^2}\right)R_- = 0. \tag{35b}$$

The eigenfunctions R_+ and R_- are orthogonal. Note that for $J = 0$, the wave functions ϕ and H_0 coincide with R_+ and $-R_-$, respectively, thus verifying the consistency of equations (35*a*–*b*), which in this case reduce to equations (33*a*–*b*). Equations (35*a*–*b*) have the form usual for the three-dimensional harmonic oscillator, albeit with a complicated energy term. It is straightforward to show that the eigenvalues E_+ of equation (35*a*) satisfy

$$(E_+^2 - m^2c^4) + \hbar\omega\sqrt{m^2c^4 + 4J(J+1)E_+^2} = (2N+3)\,\hbar\omega mc^2 \tag{36a}$$

with the principal quantum number N being a positive integer, whereas those of equation (35*b*) (denoted E_-) follow

$$(E_-^2 - m^2c^4) - \hbar\omega\sqrt{m^2c^4 + 4J(J+1)E_-^2} = (2N+3)\hbar\omega mc^2. \tag{36b}$$

The solutions of the nonlinear eigenvalue equations (36*a*–*b*) take the form

$$\frac{1}{2mc^2}(E_\pm^2 - m^2c^4) = (N + \tfrac{3}{2})\hbar\omega + J(J+1)\frac{(\hbar\omega)^2}{mc^2} \mp \Delta \tag{37a}$$

where

$$\Delta = \hbar\omega\left(J + \tfrac{1}{2}\right)\left(1 + \frac{a_1}{a_0}\frac{\hbar\omega}{mc^2} + \frac{a_2}{a_0}\left(\frac{\hbar\omega}{mc^2}\right)^2\right)^{1/2} \tag{37b}$$

with $a_0 = (2J+1)^2$, $a_1 = 4J(J+1)(2N+3)$ and $a_2 = 4J^2(J+1)^2$.

Note that for $J = 0$, the $E_\pm$ eigenvalues agree exactly with those that one would obtain directly from equations (33*a*–*b*), which describe two decoupled one-dimensional oscillators with different zero-point motions; this verifies the consistency of our result.

As shown in equation (37*a*), the energy of the DKP oscillator in unnatural-parity states involves the usual three-dimensional harmonic oscillator energy, a second term proportional to $J(J+1)$ which appears as some kind of rotational energy and a third energy contribution Δ which is a complicated function of the oscillator frequency, J and N with no obvious physical interpretation.

In the limit where the oscillator frequencies are such that $\hbar\omega \ll mc^2$, keeping only the first-order term in ω in equations (37) leads to

$$\frac{1}{2mc^2}(E_+^2 - m^2c^4) \equiv \epsilon^+_{\text{n.r.}} \simeq (N - J + 1)\hbar\omega \tag{38a}$$

$$\frac{1}{2mc^2}(E_-^2 - m^2c^4) \equiv \epsilon^-_{\text{n.r.}} \simeq (N + J + 2)\hbar\omega. \tag{38b}$$

This shows that our solutions have the correct non-relativistic limits since the levels in equations (38) are those of a usual 3D non-relativistic oscillator with a spin-orbital coupling of strength $-\hbar\omega$. In this limit, they could have also been obtained directly from equation (16). Furthermore, taking this limit suggests the interpretation of the E_+ and E_- energies as 'spin-orbit partners', E_+ being associated with $J = L + 1$ and E_- with $J = L - 1$. This is best illustrated in figure 1 which shows, for fixed values of N and J, the variations of the relativistic and non-relativistic eigenenergies with $\hbar\omega/mc^2$.

The unnatural-parity E_+ levels for $N \leqslant 9$ are presented in figure 2 alongside the $\epsilon^+_{\text{n.r.}}$ and the $(N + \frac{3}{2})\hbar\omega$ levels for reference. The non-relativistic spectrum is simple: the levels

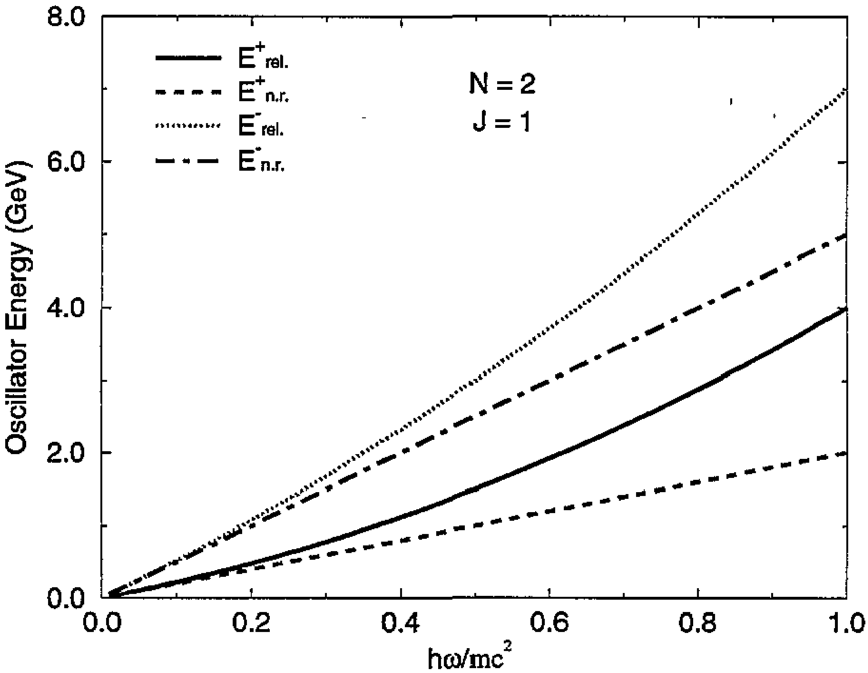

Figure 1. Variation of the DKP and non-relativistic oscillator energies with ω.

are equidistant, separated by $2\hbar\omega$, and increase in energy as $N - J$ increases from 1 to 9. For a given value of $N - J$, all the energy levels (N, J^{π}) (with $N - J$ odd and $J < N$) are infinitely degenerate. We call these levels with a definite $N - J$ the oscillator shell. This accidental degeneracy is not present in the exact DKP-oscillator E_+ eigenspectrum which displays only a $(2J + 1)$ degeneracy arising from its rotational invariance. The eigenstates are found to cluster in several groups of states belonging to fixed values of $N - J$, i.e. they are associated with the same oscillator shell in the non-relativistic limit. Within each band of states, the levels decrease in energy with increasing total angular momentum J, the bandheads being the $(N, 0^-)$ states. These properties are analogous to the situation where N (instead of $N - J$) is the quantum number that specifies the degenerate levels of a non-relativistic harmonic oscillator or the major shells of the nuclear shell model. Note that unlike $J > 0$ which have lower energies than their non-relativistic counterparts, the $(N, 0^-)$ bandheads have the same energy as their non-relativistic analogues and increase in energy (as do the bands) with increasing $N - J$.

The E_- eigenspectrum is now presented in figure 3 together with the non-relativistic $\epsilon^-_{\text{n.r.}}$ energy levels for $N \leqslant 9$. The latter are $2\hbar\omega$-equidistant and increase as $N+J$ increases from 1 up to its maximum value of 17. While all the non-relativisitic (N, J^{π}) levels associated with the same $N + J$ oscillator shell (with $N + J$ odd and $J < N$) are degenerate, with a finite degeneracy in this case, their relativistic analogues are not. The exact DKP oscillator states are found to cluster into bands of states belonging to specific values of N + J. Within all the major shells, the energy levels rise with increasing J starting from the $(N, 0^-)$ bandhead states. The $(N, 0^-)$ bandheads, which are equidistant in energy, coincide with their non-relativistic counterparts whereas the (N, J^{π}) states lie at higher energies than their non-relativistic analogues.

For a more quantitative analysis of the bands in the DKP oscillator spectra, we first show in figure 4 the energies of the $N - J = 1$ band as a function of $J(J+1)$ for different values of the oscillator frequency. For a given frequency, the levels within this major shell, as well as the spacings between them, are seen to decrease with increasing angular momentum. For larger oscillator frequencies, the energies for any given angular momentum and the gaps between adjoining levels rise but the overall band pattern of decreasing oscillator energies with increasing angular momenta is maintained. Figure 5 displays different $N - J$ bands for a fixed oscillator frequency as a function of $J(J + 1)$. Here also the major shells have the same energy decaying with increasing J behaviour although for any given value of J, the energies and the intervals between adjacent levels increase as $N - J$ increases.

We now turn to the second class of bands of this DKP oscillator. As an example, figure 6 plots all E_- energy levels belonging to the $N + J = 49$ band versus $J(J + 1)$ for different oscillator frequencies. (Since the pattern is the same for all major shells, this large value of $N + J$ is chosen to involve a large number of states.) It is indeed remarkable that the DKP oscillator energies constitute nearly perfect rotational bands. There are deviations from the rotational patterns at low angular momenta. These single-particle rotational bands are of the finite type since for $N + J$ fixed, they terminate at some J_{max}. The effective rotational moments of inertia are sensitive to the oscillator frequencies since the slopes of the bands are found to vary substantially with increasing ω. Figure 7 alternately represents the energies of five different $N + J$ bands for a specific oscillator frequency as a function of $J(J + 1)$. The DKP oscillator energies now lie on rotational bands whose slopes hardly change with $N + J$. This implies that the effective rotational moments of inertia are rigid and insensitive to such variations.

Of course, it should be pointed out that these rotational bands are unlike the usual ones in which the levels are associated with the same intrinsic motion but different angular momenta.

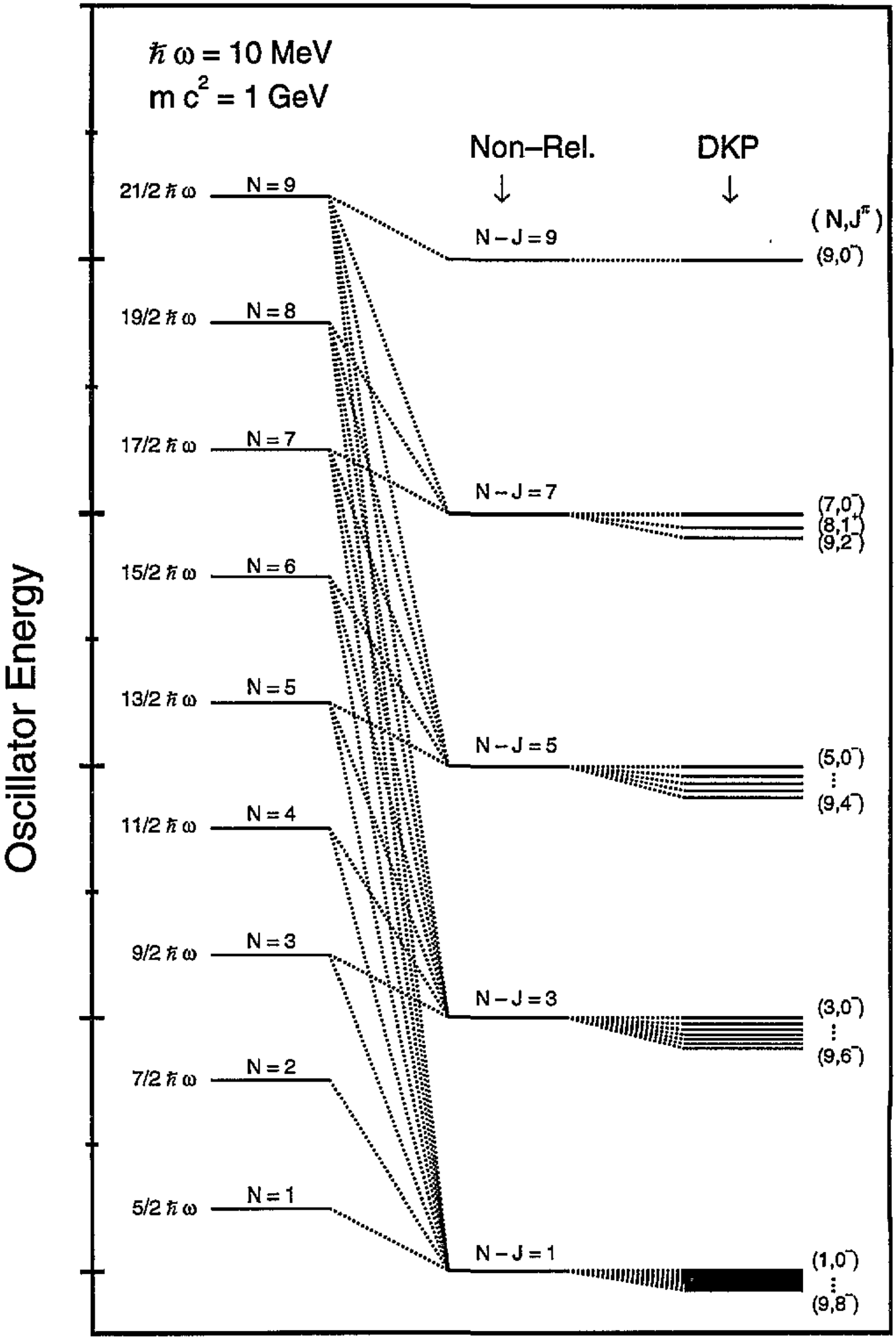

Figure 2. DKP and non-relativistic spectra associated with $J = L + 1$ for $N \leqslant 9$. The E_+ DKP-oscillator levels are on the right, non-relativistic $\epsilon^+_{\rm n.r.}$ energies lying in the centre. The $(N + \frac{3}{2})\hbar\omega$ states on the left are given for reference. The dotted lines between the DKP and non-relativistic oscillator levels link states with the same quantum numbers (N, J^π).

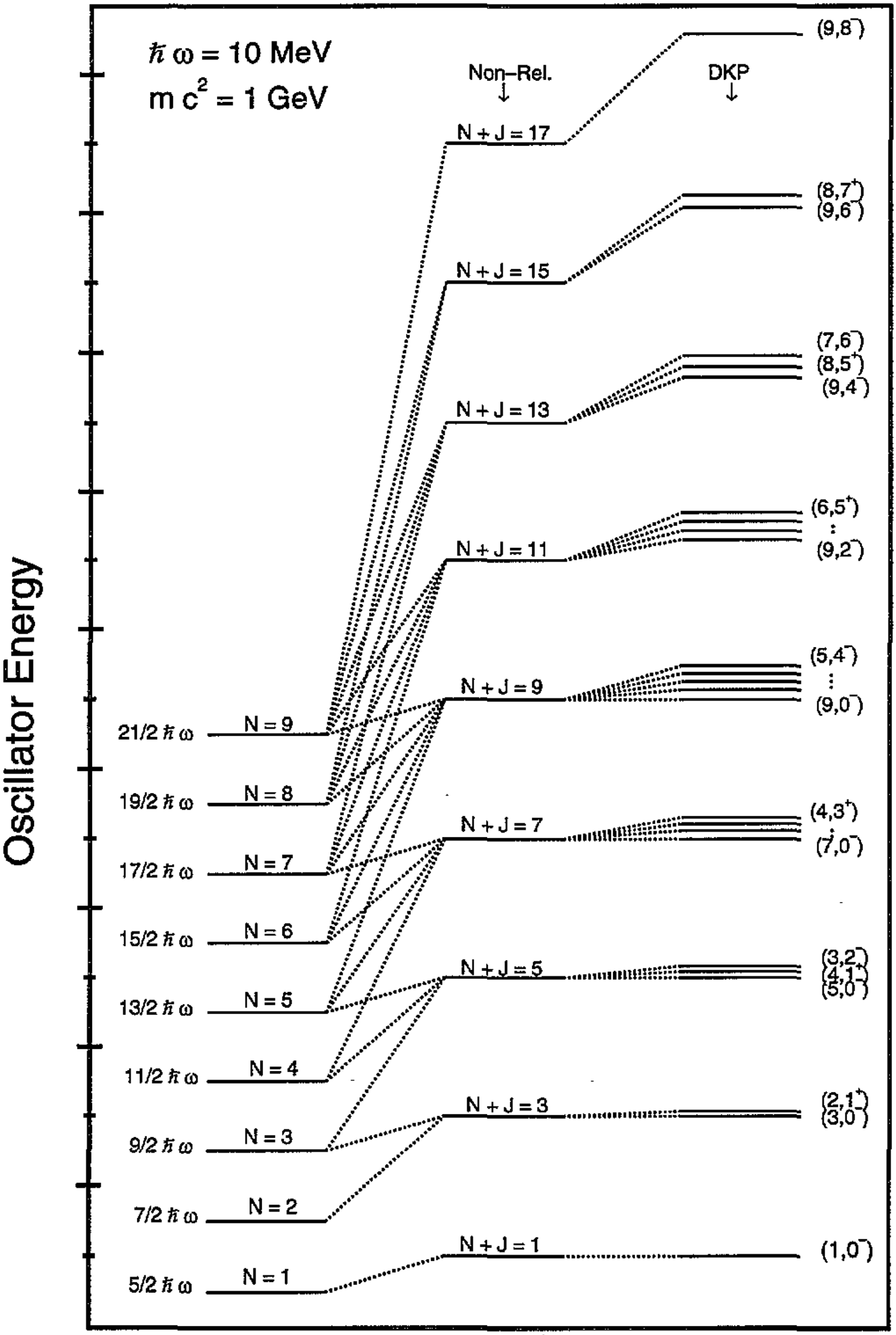

Figure 3. DKP and non-relativistic oscillator energy levels associated with $J = L - 1$ for $N \leqslant 9$.

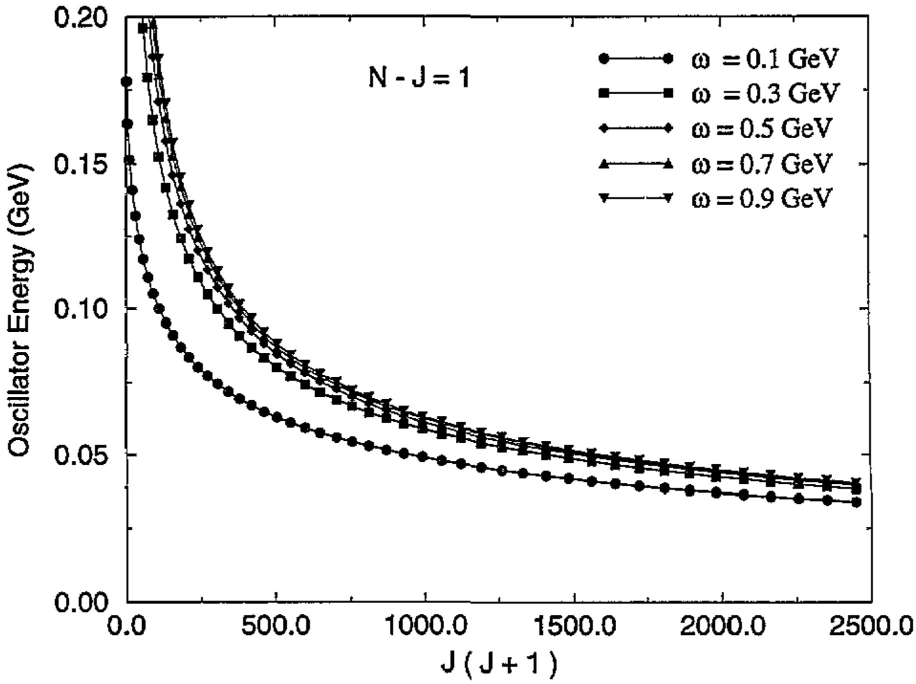

Figure 4. Energy levels of the $N - J = 1$ band as a function of $J(J+1)$ for different oscillator frequencies.

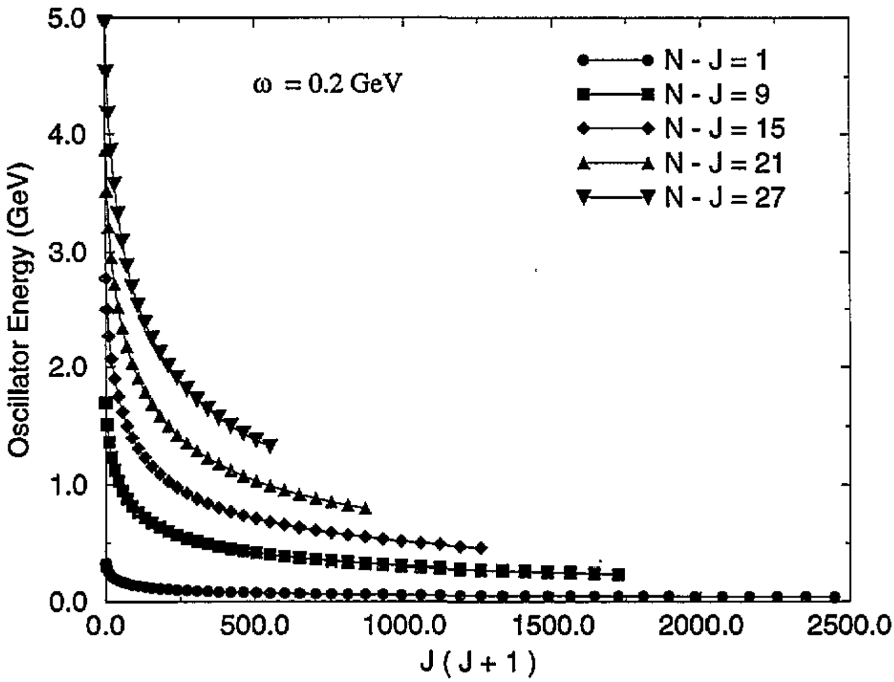

Figure 5. Energy levels of the $N - J = 1, 9, 15, 21, 27$ bands versus $J(J+1)$ for $\hbar\omega = 0.2$ GeV.

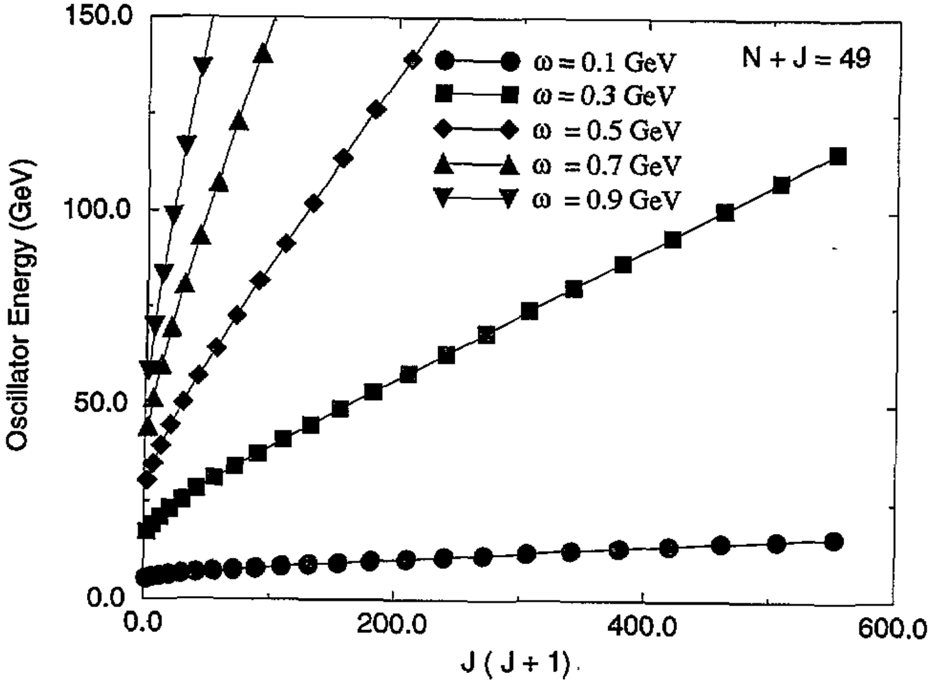

Figure 6. Energy levels of the $N + J = 49$ band as a function of $J(J+1)$ for different oscillator frequencies.

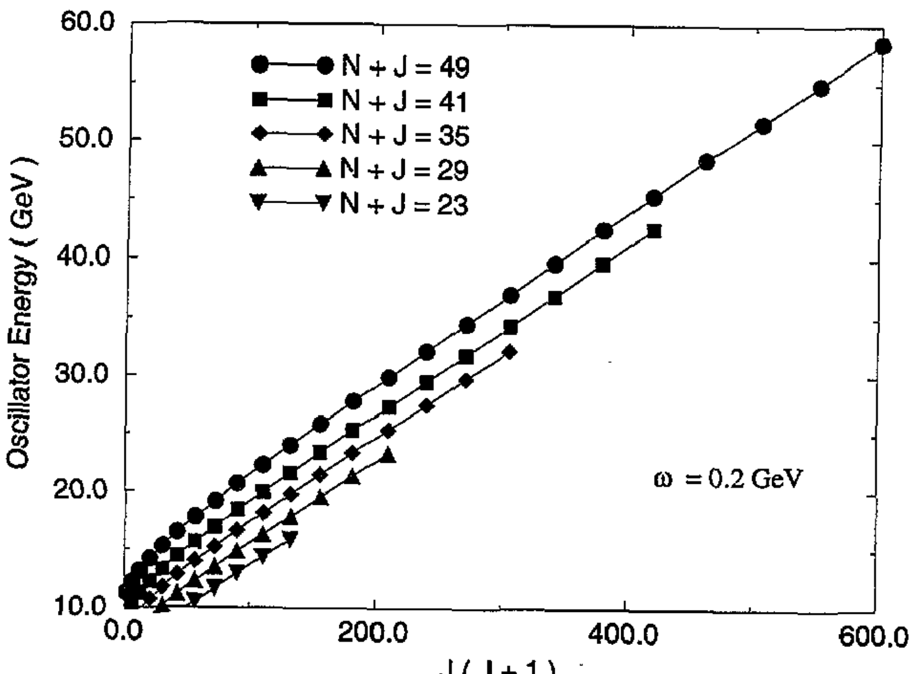

Figure 7. Energy levels of the $N + J = 23$, 29, 35, 41, 49 bands against $J(J+1)$ for $\hbar\omega = 0.2$ GeV.

Here the single particle states involve different radial as well as rotational motions. Note that this behaviour is not particularly tied to this DKP oscillator. Buck [14] also found that, when solving the Schrödinger equation for deep, bell-shaped potentials, the levels with a fixed value of $N = 2n + \ell$ (these states are degenerate for a harmonic oscillator) lie on a straight line when plotted against $\ell(\ell+1)$. Geometric arguments in terms of the shapes of the potentials which can give rise to these rotational-like bands have been put forward to explain this behaviour [14, 15].

Finally, the radial wavefunctions have to be defined so as to complete the solution. The eigenfunctions R_+ and R_- have the same form as those of the usual three-dimensional non-relativistic harmonic oscillator. All the radial components of the DKP spinor can now be determined from equations (33*c*–*d*) and (34). The normalization condition

$$\int_0^\infty \mathrm{Re}\left[F_1^*(r)G_1(r) + F_{-1}^*(r)G_{-1}(r)\right] \mathrm{d}r = \tfrac{1}{2} \tag{39}$$

has to be used to normalize the wavefunctions.

5. Conclusion

We have introduced a new potential in the DKP equation. Since in the non-relativistic limit the DKP equation of motion for scalar bosons leads to the usual harmonic oscillator and becomes a harmonic oscillator with a spin-orbit coupling of the Thomas form for vector bosons, we call the system a DKP oscillator. This oscillator is a relativistic generalization of the quantum harmonic oscillator for scalar and vector bosons. We have shown that it conserves the total angular momentum and that it is exactly solvable, and we have computed and discussed the eigensolutions for scalar and vector bosons.

The renewed interest in the Dirac oscillator has generated studies of, for instance, its group-theoretical properties [16] and hidden supersymmetry [5, 17]. Such investigations of the DKP oscillator would be most useful to gain further insight into the physical meaning of this oscillator.

Acknowledgments

We wish to thank R C Johnson for useful discussions. The financial support of SERC through GR/H53648 is gratefully acknowledged.

Appendix

Using the spin-1 representation for which

$$(s_m)_{kl} = -\mathrm{i}\varepsilon_{klm} \tag{A1}$$

ε_{klm} being the totally antisymmetric Levi-Civita symbol, we have

$$\boldsymbol{p}^+ \wedge (\boldsymbol{p}^- \wedge \boldsymbol{A}) = \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) - \boldsymbol{p}^2\boldsymbol{A} + m^2\omega^2\left(\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) - \boldsymbol{r}^2\boldsymbol{A}\right) + m\omega(2 + \boldsymbol{L}\cdot\boldsymbol{s})\boldsymbol{A} \tag{A2}$$

and

$$\boldsymbol{p}^+(\boldsymbol{p}^-\cdot\boldsymbol{A}) = \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{A}) + m^2\omega^2\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A}) - m\omega(1 + \boldsymbol{L}\cdot\boldsymbol{s})\boldsymbol{A}. \tag{A3}$$

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